Math 255A' Lecture 7 Notes

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October 11, 2019

1 The Open Mapping Theorem, Closed Graph Theorem, and Uniform Boundedness Principle

1.1 The open mapping theorem

Definition 1.1. Let X and Y be topological spaces. A function $f : X \to Y$ is **open** if $f[U]$ is open for all open $U \subseteq X$.

Remark 1.1. In metric spaces, this is equivalent to: For all $B(x, r) \subseteq X$, there is an $\varepsilon > 0$ such that $f[B(x, r)] \supseteq B(f(x), \varepsilon)$. In normed spaces, it is enough to check this at $x = 0_X$.

Theorem 1.1 (Open mapping theorem)**.** *Let X, Y be Banach spaces. If $A : X \to Y$ is a bounded linear surjection, then A is open.*

Proof. Step 1: Write $Y = \bigcup_{n=1}^\infty \overline{A(B_X(0,n))}$. By the Baire category theorem, these cannot all be nowhere dense. So there exist $n \in \mathbb{N}$, $y \in Y$, $t > 0$ such that $\overline{A(B_X(0,n))} \supseteq B_y(y,t)$. The left hand side is symmetric under $z \mapsto -z$, so $\overline{A(B_X(0,n))} \supseteq B_Y(-y,t)$, as well. By convexity,

$$\begin{aligned}\overline{A(B_X(0,n))} &\supseteq \left\{\frac{1}{2}(y+z) + \frac{1}{2}(-y+w) : \|z\|_Y, \|w\|_Y < t\right\} \\ &= \left\{\frac{1}{2}z + \frac{1}{2}w : \|z\|_Y, \|w\|_Y < t\right\} \\ &= B_Y(0,t).\end{aligned}$$

Step 2: For any $a > 0$,

$$\overline{A(B_X(0,an))} \supseteq B(0,at).$$

Step 3: We will show that $A(B_X(0,2)) \supseteq B(0,r)$. for $r = t/n$. Let $y \in B_Y(0,r)$. By step 2, there is an $x_1 \in B_X(0,1)$ such that $\|y - Ax_1\| < r/2$. Let $y_1 = y - Ax_1$, and choose $x_2 \in B_x(0,1/2)$ such that $\|y_1 - Ax_2\| < r/2$. In this way, pick y_n, x_{n+1} for each n. Let

$x = \sum_{n=1}^{\infty} x_n$; this converges because the lengths are bounded by a convergent geometric series: $\|x\| \le \sum_n \|x_n\| < 2$. Then $Ax = \sum_{n=1}^{\infty} Ax_n$. For each $N \in \mathbb{N}$,

$$y - \sum_{n=1}^{N} Ax_n = y_1 - \sum_{n=2}^{N} Ax_n = y_2 - \sum_{n=3}^{N} Ax_n = \cdots = y_N,$$

and $\|y_N\| - r/2^{N-1} \to 0$. So $y = Ax$. □

Corollary 1.1. *A bounded linear bijection between Banach spaces is an isomorphism.*

Proof. Since $A : X \to Y$ is a bijection, A^{-1} exists as a linear transformation $Y \to X$. Boundedness of A^{-1} is precisely the openness of A. □

Definition 1.2. If $A : X \to Y$, then **graph** of A is $\operatorname{gra}(A) := \{(x, Ax) : x \in X\} \subseteq X \oplus Y$. It is a linear subspace of $X \oplus Y$ with the **graph norm** $\|(x, y)\| = \|x\|_X + \|y\|_Y$.

1.2 The closed graph theorem

Corollary 1.2 (Closed graph theorem)**.** *Let X, Y be Banach spaces, and let $A : X \to Y$ be a linear transformation. If* $\operatorname{gra}(A)$ *is closed, then A is continuous.*

Proof. $\operatorname{gra}(A)$ is a closed subspace of a Banach space, so it is complete. In the following diagram, $A = P_2 \circ \tilde{A}$, so it is enough to show that $\tilde{A}$ is continuous.

$$\begin{array}{ccc} A & \xrightarrow{\tilde{A}: x \mapsto (x,\, Ax)} & \operatorname{gra}(A) \\ & \searrow^{A} & \downarrow{\scriptstyle P_2 : (x,\, y) \mapsto y} \\ & & Y \end{array}$$

But $\tilde{A} = (P_1|_{\operatorname{gra}(A)})^{-1}$, so it is continuous by the previous corollary. □

Example 1.1. Let $X = C^{(1)}[0, 1]$ and $Y = C[0, 1]$, both with the uniform norm. Then A sending $f \mapsto f'$ is not continuous. But its graph, $\operatorname{gra}(A) = \{(f, f') : f \in C^{(1)}\}$ is closed: Suppose $(f_n)_n$ is such that $f_n \to g$ uniformly, and $f_n' \to h$ uniformly. Then $f_n - f \to 0$, which means that $f_n' - g' \to h - g'$ uniformly; sp we may assume that $f_n \to 0$ and $f_n' \to h$. We must show that $h = 0$. We have that for all $t \in [0, 1]$. so

$$\int_0^t h(s)\, ds = \lim_n \int_0^t f_n' = \lim_n [f_n(t) - f_n(0)] = 0.$$

So $h = 0$.

In general, $\operatorname{gra}(A)$ is closed if $x_n \to 0$ and $Ax_n \to y$ implies $y \to 0$. This is often easier to check than continuity.

1.3 The principle of uniform boundedness

Theorem 1.2 (Principle of uniform boundedness)**.** *Let X be a Banach space, let Y be a normed space, and let $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. Assume that* $\sup\{\|Ax\| : A \in \mathcal{A}\} < \infty$ *for all $x \in X$. Then* $\sup\{\|A\| : A \in \mathcal{A}\} < \infty$*.*

Instead of citing Baire category, we will adapt the proof of that theorem to prove this.

Proof. Assume, towards a contradiction, that $M(x) := \sup\{\|Ax\| : A \in \mathcal{A}\} < \infty$ for all x, but $\sup_{A \in \mathcal{A}} \|A\| = \infty$. So for every $\varepsilon > 0$, there is an $x \in X$ and $A \in \mathcal{A}$ such that $\|x\| < \varepsilon$ and $\|Ax\| > 1/\varepsilon$.

Construct sequences (x_n) in x and (A_n) in $\mathcal{A}$ by recursion: Pick any $\|x_1\| = 1$ and any A_1. Now choose (x_2, A_2) such that $\|x_2\| \leq 1/2$, $\|A_1 x_2\| \leq 1/2$, and $\|A_2 x_2\| > 2 + M(x_1)$. Now choose (x_3, A_3) such that $\|x_3\|, \|A_1 x_3\|, \|A_2 x_3\| < 1/4$ but $\|A_3 x_3\| > 3 + M(x_1) + M(x_2)$. At the n-th stage, choose (x_n, A_n) such that $\|x_n\|, \|A_1 x_n\|, \dots, \|A_{n-1} x_n\| < 1/2^n$ but $\|A_n x_n\| ? n + M(x_1) + M(x_2) + \cdots + M(x_{n-1})$.

Now let $x = \sum_{n=1}^{\infty} x_n$. Then

$$\begin{aligned} A_k x &= \sum_{n=1}^{\infty} A_k x_n \\ &= \underbrace{\sum_{n=1}^{k-1} A_k x_n}_{\|\cdot\| \leq M(x_1) + \cdots + M(x_{k+1})} + \underbrace{A_k x_k}_{\|\cdot\| > k + M(x_1) + \cdots + M(x_{k-1})} + \underbrace{\sum_{k+1}^{\infty} A_k x_n}_{\|\cdot\| \leq 2^{-k}} \end{aligned}$$

So $\|A_k x\| > k - 1$, which implies that $M(x) = \infty$. This is a contradiction. ☐

Corollary 1.3. *Let X be a Banach space. If $A \subseteq X^*$ is such that* $\sup\{|L(x)| : L \in A\}$ *for all x, then* $\sup_{L \in A} \|L\| < \infty$*.*

Corollary 1.4. *Let Y be a normed space. If $A \subseteq Y$ and* $\sup\{|L(a)| : a \in A\} < \infty$ *for all $L \in Y^*$, then* $\sup_{a \in A} \|a\| < \infty$*.*

Proof. Consider the natural embedding of A into $\hat{A} \subseteq Y^{**}$. ☐

Corollary 1.5. *Let X be a Banach space, let Y be a normed space, and let $A \subseteq \mathcal{B}(C, Y)$. If* $\sup\{|L(Ax)| : A \in \mathcal{A}\} < \infty$ *for all $x \in X$ and $L \in Y^*$, then A is uniformly bounded.*

Proof. This is a double application of the principle of uniform boundedness. ☐

Theorem 1.3 (Banach-Steinhaus)**.** *Let X, Y be Banach spaces. Let $(A_n)_n$ be a sequence in $\mathcal{B}(X, Y)$. If for every x, there is a y such that $A_n x \to y$, then*

1. $\sup_n \|A_n\| < \infty$,
2. *There exists some $A \in \mathcal{B}(X < Y)$ such that $A_n x \to Ax$.*